ON THE K-THEORY OF ELLIPTIC CURVES

KEVIN P. KNUDSON

Abstract. Let A be the coordinate ring of an affine elliptic curve (over an infinite field k) of the form $X-\{p\}$, where X is projective and p is a closed point on X. Denote by F the function field of X. We show that the image of $H_\bullet(GL_2(A),\mathbb{Z})$ in $H_\bullet(GL_2(F),\mathbb{Z})$ coincides with the image of $H_\bullet(GL_2(k),\mathbb{Z})$. As a consequence, we obtain numerous results about the K-theory of A and X. For example, if k is a number field, we show that $r_2(K_2(A)\otimes\mathbb{Q})=0$, where r_m denotes the mth level of the rank filtration.

1. Introduction

Computing the K-theory of a scheme X is a very difficult task. Even the simplest case $X=\operatorname{Spec} k$, where k is a field, is not completely solved, although a great deal is known. The next case to consider is when X is a curve over k, and it is here that the complexity grows rapidly. Some curves of genus zero present no real difficulty thanks to the fundamental theorem: $K_i(R[t,t^{-1}])=K_i(R)\oplus K_{i-1}(R)$ for R regular. The K-theory of elliptic curves, on the other hand, has proved to be much more elusive.

A great deal of recent work has focused on the construction of specific elements in the K-theory of elliptic curves, particularly in the second group K_2. This program goes back to the work of S. Bloch [2], who constructed a regulator map on K_2 and used it to find nontrivial elements. A. Beilinson [1] generalized this construction and made a number of conjectures relating the dimension of $K_2\otimes\mathbb{Q}$ with the values of L-functions on the curve. More recently, Goncharov–Levin [6], Rolshausen–Schappacher [10], and Wildeshaus [15] have made further progress in this area.

In this paper we consider the following situation. Let E be an affine elliptic curve defined by the Weierstrass equation $F(x,y)=0$, where

$$F(x,y)=y^2+a_1xy+a_3y-x^3-a_2x^2-a_4x-a_6.$$

Here, the a_i lie in an infinite field k. Denote by $\overline{E}$ the projective curve $E\cup\{\infty\}$ and by F the function field of $\overline{E}$. Denote by A the affine coordinate ring of E; it is a Dedekind domain with field of fractions F. We have $A^\times=k^\times$.

Date: July 13, 1998, revised September 22, 1998.
1991 *Mathematics Subject Classification.* 19D55, 20G10.
Supported by an NSF Postdoctoral Fellowship, grant no. DMS–9627503.

Consider the obvious embedding $i : GL_2(A) \longrightarrow GL_2(F)$. The main result of this paper is the following.

Theorem 1.1. *The image of the map*

$$i_* : H_\bullet(GL_2(A), \mathbb{Z}) \longrightarrow H_\bullet(GL_2(F), \mathbb{Z})$$

coincides with the image of

$$(i|_{GL_2(k)})_* : H_\bullet(GL_2(k), \mathbb{Z}) \longrightarrow H_\bullet(GL_2(F), \mathbb{Z}).$$

This is a consequence of an explicit computation of the homology of $PGL_2(A)$ due to the author [7] (recalled in Section 4 below). The proof of Theorem 1.1 is given in Section 6.

Remark. Theorem 1.1 and its corollaries in Sections 2 and 3 are valid also for singular cubic curves $F(x, y) = 0$. We shall point out the necessary modifications needed to prove this below.

From this result we deduce a number of facts about the K-theory of E and $\overline{E}$. Recall the *rank filtration* of the rational K-theory $K_\bullet(R)_\mathbb{Q} := K_\bullet(R) \otimes_\mathbb{Z} \mathbb{Q}$ of a ring R:

$$r_m K_n(R)_\mathbb{Q} = \operatorname{im}\{H_n(GL_m(R), \mathbb{Q}) \longrightarrow H_n(GL(R), \mathbb{Q})\} \cap K_n(R)_\mathbb{Q}.$$

Corollary 1.2. *The image of the map $r_2K_n(A)_\mathbb{Q} \longrightarrow r_2K_n(F)_\mathbb{Q}$ coincides with the image of $r_2K_n(k)_\mathbb{Q} \longrightarrow r_2K_n(F)_\mathbb{Q}$.*

In particular, when $n = 2$ we see that the image of $r_2K_2(A)_\mathbb{Q} \longrightarrow r_2K_2(F)_\mathbb{Q}$ coincides with the image of $K_2(k)_\mathbb{Q}$.

Remark. This corollary is valid for *any* field k. Indeed, if k is finite, then the rational homology $H_\bullet(GL_2(A), \mathbb{Q})$ vanishes in positive degrees (as does $H_\bullet(GL_2(k), \mathbb{Q})$) from which it follows that $r_2K_n(A)_\mathbb{Q} = 0$.

Define a filtration $r_\bullet K_\bullet(\overline{E})_\mathbb{Q}$ by pulling back the rank filtration of $K_\bullet(A)_\mathbb{Q}$:

$$r_m K_n(\overline{E})_\mathbb{Q} := (f^*)^{-1}(r_m K_n(A)_\mathbb{Q}),$$

where $f : E \longrightarrow \overline{E}$ is the inclusion and $f^* : K_\bullet(\overline{E}) \longrightarrow K_\bullet(E) = K_\bullet(A)$ is the induced map in K-theory. Then we obviously have the following result.

Corollary 1.3. *The image of $r_2K_n(\overline{E})_\mathbb{Q} \longrightarrow r_2K_n(F)_\mathbb{Q}$ coincides with the image of $r_2K_n(k)_\mathbb{Q} \longrightarrow r_2K_n(F)_\mathbb{Q}$.*

We study the filtration $r_\bullet$ in greater detail in Section 2. In Section 3 we specialize to the case where k is a number field. In this case, we show that $r_2K_2(A)_\mathbb{Q} = 0$.

In the case $n = 2$, results of Nesterenko–Suslin [9] imply that $r_3K_2(A)_\mathbb{Q} = K_2(A)_\mathbb{Q}$. A description of the homology of $PGL_3(A)$ (or $GL_3(A)$) would provide a great deal of insight into the structure of $K_2(\overline{E})_\mathbb{Q}$, especially over a number field. Such a computation remains elusive, however.

Acknowledgements. I thank Dick Hain for practically insisting that I study this question. Andrei Suslin provided valuable insight regarding the proof of Theorem 1.1. Thanks also go to Mark Walker for answering several (perhaps silly) questions.

2. The Rank Filtration

The rational K-groups of affine schemes admit the rank filtration mentioned in the introduction. Since $BGL(R)^+$ is an H-space, the Milnor–Moore Theorem [8] implies that the Hurewicz map

$$h : \pi_\bullet(BGL(R)^+) \otimes \mathbb{Q} \longrightarrow H_\bullet(GL(R), \mathbb{Q})$$

is injective with image equal to the primitive elements of the homology. The rank filtration is the increasing filtration defined by

$$r_m K_n(R)_\mathbb{Q} = \mathrm{im}\{H_n(GL_m(R), \mathbb{Q}) \longrightarrow H_n(GL(R), \mathbb{Q})\} \cap K_n(R)_\mathbb{Q}.$$

By Theorem 2.7 of [9], the map $H_2(GL_3(A), \mathbb{Z}) \to H_2(GL(A), \mathbb{Z})$ is surjective so that $r_3K_2(A)_\mathbb{Q} = K_2(A)_\mathbb{Q}$. The rank filtration of $K_2(A)_\mathbb{Q}$ then has the form

$$0 = r_1K_2(A)_\mathbb{Q} \subseteq r_2K_2(A)_\mathbb{Q} \subseteq r_3K_2(A)_\mathbb{Q} = K_2(A)_\mathbb{Q}$$

(the vanishing of r_1 is a consequence of the vanishing of $r_1K_2(k)_\mathbb{Q}$ for infinite fields [9], and the fact that $A^\times = k^\times$).

Define an increasing filtration $r_\bullet$ of $K_n(\overline{E})_\mathbb{Q}$ as follows. Let $f : E \longrightarrow \overline{E}$ be the canonical inclusion and denote by f^* the induced map on K-theory. We define $r_mK_n(\overline{E})_\mathbb{Q}$ by

$$r_mK_n(\overline{E})_\mathbb{Q} = (f^*)^{-1}(r_mK_n(A)_\mathbb{Q}).$$

There is a commutative diagram

$$\begin{array}{ccc} r_mK_n(\overline{E})_\mathbb{Q} & \longrightarrow & r_mK_n(A)_\mathbb{Q} \\ & \searrow & \downarrow \\ & & r_mK_n(F)_\mathbb{Q}. \end{array} \tag{1}$$

Proposition 2.1. *The image of $r_2K_n(A)_\mathbb{Q} \longrightarrow r_2K_n(F)_\mathbb{Q}$ coincides with the image of $r_2K_n(k)_\mathbb{Q} \longrightarrow r_2K_n(F)_\mathbb{Q}$.*

Proof. By Theorem 1.1, the image of

$$i_* : H_\bullet(GL_2(A), \mathbb{Z}) \longrightarrow H_\bullet(GL_2(F), \mathbb{Z})$$

coincides with the image of $(i|_{GL_2(k)})_*$. Consider the commutative diagram

$$\begin{array}{ccc} H_n(GL_2(A), \mathbb{Q}) & \longrightarrow & H_n(GL(A), \mathbb{Q}) \\ \downarrow & & \downarrow \\ H_n(GL_2(F), \mathbb{Q}) & \longrightarrow & H_n(GL(F), \mathbb{Q}). \end{array}$$

It follows that the image of $H_n(GL_2(A), \mathbb{Q})$ in $H_n(GL(F), \mathbb{Q})$ coincides with the image of $H_n(GL_2(k), \mathbb{Q})$; *i.e.,* the image of $r_2K_n(A)_\mathbb{Q} \to r_2K_n(F)_\mathbb{Q}$ coincides with the image of $r_2K_n(k)_\mathbb{Q}$. □

Corollary 2.2. *The image of $r_2K_n(\overline{E})_\mathbb{Q} \longrightarrow r_2K_n(F)_\mathbb{Q}$ coincides with the image of $r_2K_n(k)_\mathbb{Q}$.*

Proof. This follows by considering the diagram (1). □

3. The Number Field Case

Suppose that the ground field k is a number field. By localizing the projective curve at its generic point we obtain the following exact sequence for K_2

$$0 \longrightarrow K_2(\overline{E})_{\mathbb{Q}} \longrightarrow K_2(F)_{\mathbb{Q}} \xrightarrow{\mathcal{T}} \bigoplus_P K_1(k(P))_{\mathbb{Q}}$$

where P varies over the closed points of $\overline{E}$ and $k(P)$ is the residue field at P. The map $\mathcal{T}$ is the *tame symbol* (see, *e.g.,* [10]).

Remark. It is not known for a single curve if $K_2(\overline{E})_{\mathbb{Q}}$ is finite dimensional. Beilinson has conjectured that the dimension of this space is related to special values of L-functions on $\overline{E}$. This conjecture was modified by Bloch and Grayson [3] to predict that the dimension is the number of infinite places of k plus the number of primes $\mathfrak{p} \subset \mathcal{O}_k$ where $\overline{E}$ has split multiplicative reduction modulo $\mathfrak{p}$. For a discussion of this see, for example, [10].

We also have the localization sequence for A:

$$\cdots \to K_{i+1}(F) \to \bigoplus_{\mathfrak{p}\text{ maximal}} K_i(A/\mathfrak{p}) \to K_i(A) \to K_i(F) \to \cdots .$$

Since $A/\mathfrak{p}$ is a finite extension of k for all $\mathfrak{p}$, the groups $K_{2m}(A/\mathfrak{p})$ are torsion. It follows that we have an exact sequence

$$0 \longrightarrow K_{2m}(A)_{\mathbb{Q}} \longrightarrow K_{2m}(F)_{\mathbb{Q}} \longrightarrow \bigoplus_{\mathfrak{p}} K_{2m-1}(A/\mathfrak{p})_{\mathbb{Q}}.$$

Proposition 3.1. *If the ground field k is a number field, then the map $K_2(\overline{E})_{\mathbb{Q}} \to K_2(A)_{\mathbb{Q}}$ is injective.*

Proof. This follows by considering the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K_2(A)_{\mathbb{Q}} & \longrightarrow & K_2(F)_{\mathbb{Q}} \\ & & \uparrow & & || \\ 0 & \longrightarrow & K_2(\overline{E})_{\mathbb{Q}} & \longrightarrow & K_2(F)_{\mathbb{Q}}. \end{array}$$

□

Proposition 3.2. *If k is a number field, then $r_2K_2(A)_{\mathbb{Q}} = 0 = r_2K_2(\overline{E})_{\mathbb{Q}}$.*

Proof. The map $K_2(A)_{\mathbb{Q}} \to K_2(F)_{\mathbb{Q}}$ is injective. But by Proposition 2.1, the image of $r_2K_2(A)_{\mathbb{Q}}$ coincides with the image of $r_2K_2(k)_{\mathbb{Q}} = K_2(k)_{\mathbb{Q}} = 0$. □

As a consequence we see that any nontrivial elements of $K_2(A)_{\mathbb{Q}}$ (and hence of $K_2(\overline{E})_{\mathbb{Q}}$) must come from $H_2(GL_3(A), \mathbb{Q})$. Thus, to prove that $K_2(\overline{E})_{\mathbb{Q}}$ is a finite dimensional vector space, it suffices to show that the image of $H_2(GL_3(A), \mathbb{Q})$ in $H_2(GL_3(F), \mathbb{Q}) = H_2(GL(F), \mathbb{Q})$ is finite dimensional.

4. The Homology of $PGL_2(A)$

The remainder of the paper is devoted to the proof of Theorem 1.1. We begin by recalling the calculation of $H_\bullet(PGL_2(A), \mathbb{Z})$ given in [7]. The proof uses the action of $PGL_2(A)$ on a certain Bruhat–Tits tree $\mathcal{X}$.

We use the description of $\mathcal{X}$ given by Takahashi [14]. Recall that A is the coordinate ring of the affine curve E with function field F. The field F has transcendence degree 1 over k and is equipped with the discrete valuation at ∞, v_∞. Denote by $\mathcal{O}_\infty$ the valuation ring and by $t = x/y$ the uniformizer at ∞. Denote by $\mathcal{L}$ the field of Laurent series in t and let v be the valuation on $\mathcal{L}$ defined by $v(\sum_{n\geq n_0} a_n t^n) = n_0$. The ring A can be embedded in $\mathcal{L}$ in such a way that $v(x) = -2$ and $v(y) = -3$; we identify A with its image in $\mathcal{L}$. Note that this embedding induces an embedding $F \to \mathcal{L}$ and that the completion of F with respect to v_∞ is $\mathcal{L}$. We therefore have a commutative diagram

$$\begin{array}{ccc} GL_2(A) & \longrightarrow & GL_2(F) \\ & \searrow & \downarrow \\ & & GL_2(\mathcal{L}). \end{array}$$

Let $G = GL_2(\mathcal{L})$ and $K = GL_2(k[[t]])$. Denote by Z the center of G. The Bruhat–Tits tree $\mathcal{X}$ is defined as follows. The vertex set of $\mathcal{X}$ is the set of cosets G/KZ. Two cosets g_1KZ and g_2KZ are adjacent if

$$g_1^{-1}g_2 = \begin{pmatrix} t & b \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{modulo}\, KZ$$

for some $b \in k$. The graph $\mathcal{X}$ is a tree [12], p. 70. Note that $GL_2(A)$ acts on $\mathcal{X}$ without inversion and that the center of $GL_2(A)$ (which is equal to $k^\times$) acts trivially on $\mathcal{X}$. It follows that the quotient $PGL_2(A)\backslash\mathcal{X}$ is defined. We describe a fundamental domain $\mathcal{D} \subset \mathcal{X}$ for the action (*i.e.*, $\mathcal{D} \cong PGL_2(A)\backslash\mathcal{X}$).

If $f_1, f_2 \in \mathcal{L}$, denote by $\phi(f_1, f_2)$ the vertex $\begin{pmatrix} f_1 & f_2 \\ 0 & 1 \end{pmatrix} KZ$. Denote by $F_x(l,m)$ and $F_y(l,m)$ the partial derivatives at (l,m) of the Weierstrass equation $F(x,y)$. Define two sets E_1 and E_2 as follows:

$$E_1 = \{(l,m) : F(l,m) = 0 \text{ and } F_y(l,m) = 0\} \cup \{\infty\}$$

and

$$E_2 = \{(l,m) : F(l,m) = 0 \text{ and } F_y(l,m) \neq 0\}.$$

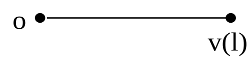

FIGURE 1. $F(l,y)=0$ has no rational solutions

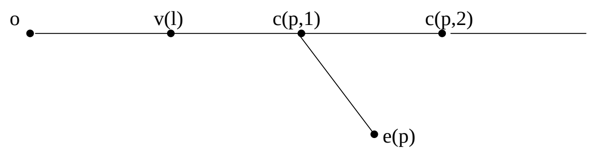

FIGURE 2. $F(l,y)=0$ has a unique rational solution

Observe that $\overline{E}=E_1\cup E_2$. Define vertices of $\mathcal{X}$ by

$$\begin{aligned} o &= \phi(t,t^{-1}); \\ v(l) &= \begin{cases}\phi(t^2,t^{-1}+lt) & \text{if } l\in k\\ \phi(1,t^{-1}) & \text{if } l=\infty;\end{cases} \\ c(p,n) &= \begin{cases}\phi(t^{n+2},\frac{y-m}{x-l}) & \text{if } p=(l,m)\in E\\ \phi(t^{-n},0) & \text{if } p=\infty;\end{cases} \\ e(p) &= \begin{cases}\phi(t^4,\frac{y-m}{x-l}+\frac{F_x(l,m)}{y-m}) & \text{if } p=(l,m)\in E_1\\ \phi(1,0) & \text{if } p=\infty.\end{cases}\end{aligned}$$

We are now ready to describe the subgraph $\mathcal{D}$. For each $l\in k\cup\{\infty\}$, the vertex $v(l)$ is adjacent to o. Denote by $\mathcal{D}(l)$ the connected component of $\mathcal{D}-\{o\}$ which contains $v(l)$. The $\mathcal{D}(l)$ fall into three types.

(1) Suppose $F(x,y)=0$ has no rational solution with $x=l$. Then $\mathcal{D}(l)$ consists only of $v(l)$ (see Figure 1).

(2) Suppose $l=\infty$ or $F(x,y)=0$ has a unique rational solution with $x=l$. Let p be the point at infinity of E or the rational point corresponding to the solution. Note that p is a point of order 2. Then $\mathcal{D}(l)$ consists of an infinite path $c(p,1),c(p,2),\dots$ and an extra vertex $e(p)$ (see Figure 2).

(3) Suppose $F(x,y)=0$ has two different solutions such that $x=l$. Let p,q be the corresponding points on E. Then $\mathcal{D}(l)$ consists of two infinite paths $c(p,1),c(p,2),\dots$ and $c(q,1),c(q,2),\dots$ (see Figure 3).

The infinite path $c(p,1),c(p,2),\dots$ is called a *cusp*. Note that there is a one-to-one correspondence between cusps and the rational points of $\overline{E}$.

Theorem 4.1 (Takahashi)**.** *The graph $\mathcal{D}$ is a fundamental domain for the action of $GL_2(A)$ on $\mathcal{X}$ (and hence is also a fundamental domain for the action of $PGL_2(A)$).* □

Remark. The theorem is true also for singular curves C given by $F(x,y)=0$ with the following modification. If the curve is singular at $p=(l,m)$, then the vertex $e(p)$ is the same as $c(p,2)$. In this case, then, the tree $\mathcal{D}(l)$ consists only of the cusp $c(p,1),c(p,2),\dots$. The proofs of the following results for

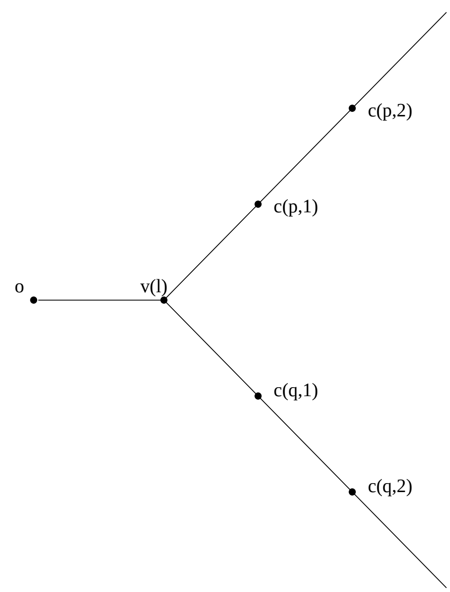

FIGURE 3. $F(l, y) = 0$ has two distinct solutions

C then go through unchanged except that the summands in the homology decomposition of $H_\bullet(PGL_2(k[C]), \mathbb{Z})$ corresponding to singular points are $H_\bullet(k^\times, \mathbb{Z})$ instead of $H_\bullet(PGL_2(k), \mathbb{Z})$.

Since $\mathcal{X}$ is contractible, we have a spectral sequence with E^1–term

$$E^1_{p,q} = \bigoplus_{\sigma^{(p)} \subset \mathcal{D}} H_q(\Gamma_\sigma, \mathbb{Z}) \Longrightarrow H_{p+q}(PGL_2(A), \mathbb{Z})$$

where Γ_σ is the stabilizer of the p–simplex σ in $PGL_2(A)$. We shall discuss the stabilizers in detail in the next section. For the purpose of computing homology, the next result is sufficient (see [14], Theorem 5). If $F(l, y) = 0$ has no rational solution, denote by $k(\omega)$ the quadratic extension of k in which $F(l, \omega) = 0$.

Proposition 4.2. *Up to isomorphism, the stabilizers* Γ_σ *are as follows:*

$$\begin{aligned}
\Gamma_o &= \{1\} \\
\Gamma_{v(l)} &\cong \begin{cases} k(\omega)^\times / k^\times & \text{in case (1)} \\ k & \text{in case (2)} \\ k^\times & \text{in case (3)} \end{cases} \\
\Gamma_{c(p,n)} &\cong \left\{ \begin{pmatrix} p & v \\ 0 & q \end{pmatrix} : p, q \in k^\times, v \in k^n \right\} / k^\times \\
\Gamma_{e(p)} &\cong PGL_2(k).
\end{aligned}$$

The stabilizer of an edge is the intersection of its vertex stabilizers (one of which is contained in the other). ∎

By Theorem 1.11 of [9], the inclusion of the diagonal subgroup into $\Gamma_{c(p,n)}$ induces an isomorphism in homology. This leads to the proof of the following, which is the main result of [7].

Theorem 4.3. *For all $i \geq 1$,*

$$\begin{aligned} H_i(PGL_2(A),\mathbb{Z}) \quad \cong & \bigoplus_{\substack{l\in k\cup\{\infty\} \\ F(l,y)=0 \text{ has unique sol.}}} H_i(PGL_2(k),\mathbb{Z}) \\ & \oplus \bigoplus_{\substack{l\in k \\ F(l,y)=0 \text{ has two sol.}}} H_i(k^\times,\mathbb{Z}) \\ & \oplus \bigoplus_{\substack{l\in k \\ F(l,y)=0 \text{ has no sol.}}} H_i(k(\omega)^\times/k^\times,\mathbb{Z}). \quad \square \end{aligned}$$

Remark. This theorem holds also in degrees ≤ 2 if k is a finite field with at least 4 elements. For in this case, the inclusion of the diagonal subgroup into $\Gamma_{c(p,n)}$ induces a homology isomorphism in degrees ≤ 2; see [11], p. 204.

The isomorphism is induced by the inclusion of the various $\Gamma_{v(l)}$ and $\Gamma_{e(p)}$. In the next section, we shall compute the image of the map

$$H_\bullet(PGL_2(A),\mathbb{Z}) \longrightarrow H_\bullet(PGL_2(F),\mathbb{Z}).$$

5. The Map $H_\bullet(PGL_2(A),\mathbb{Z}) \to H_\bullet(PGL_2(F),\mathbb{Z})$

To compute the image of $H_\bullet(PGL_2(A),\mathbb{Z})$ in $H_\bullet(PGL_2(F),\mathbb{Z})$, we must examine the various Γ_v in greater detail. If $p=\infty$, then the stabilizer $\Gamma_{e(\infty)}$ is the subgroup $PGL_2(k)$ of $PGL_2(A)$. Hence, under the map $j : PGL_2(A) \to PGL_2(F)$, $\Gamma_{e(\infty)}$ maps to $PGL_2(k) \subset PGL_2(F)$.

The other stabilizers for $l \neq \infty$ are *not* subgroups of $PGL_2(k)$, although they are isomorphic to such. We have the following result.

Theorem 5.1. *For each $l \in k$, the stabilizers $\Gamma_{v(l)}$ and $\Gamma_{e(p)}$ ($p=(l,m)$) are conjugate in $PGL_2(F)$ to subgroups of $PGL_2(k)$.*

Corollary 5.2. *The image of $j_* : H_\bullet(PGL_2(A),\mathbb{Z}) \to H_\bullet(PGL_2(F),\mathbb{Z})$ coincides with the image of $H_\bullet(PGL_2(k),\mathbb{Z})$.*

Proof. It is well-known (see [5], p. 48) that conjugation induces the identity on homology. It follows that if H_1, H_2 are conjugate subgroups of a group G, then the images of $H_\bullet(H_i,\mathbb{Z}) \to H_\bullet(G,\mathbb{Z})$ coincide. Since each stabilizer which appears in the homology decomposition of $PGL_2(A)$ is conjugate in $PGL_2(F)$ to a subgroup of $PGL_2(k)$, the result follows. $\square$

Proof of Theorem 5.1. To keep the notation as simple as possible, we only prove the case $\Gamma_{v(0)}$ and in the case $F(0,0)=0=F_y(0,0)$, $\Gamma_{e(0,0)}$. All other

cases are similar (but notationally more complex). For $r_1, \ldots, r_4 \in k$ define

$$M_2(r_1, r_2) = \begin{pmatrix} r_2 y + r_1 & -r_2\left(\frac{y^2 + a_3 y - a_6}{x}\right) \\ r_2 x & -r_2 y - a_3 r_2 + r_1 \end{pmatrix}$$

and

$$M_4(r_1, r_2, r_3, r_4) = \left(\begin{array}{c|c} r_4 xy + r_3(x^2 + a_2 x + a_4) & -r_4 y^2 - r_3 y(x + a_2) + a_4 r_4 (x + a_2) \\ +r_2 y + r_1 & -r_2(x^2 + a_2 x + a_4 - a_1 y) \\ \hline r_4 x^2 + r_3(y + a_1 x) & -r_4 xy - r_3(x^2 + a_2 x + a_4) \\ +r_2 x + a_4 r_4 & -r_2 y + a_1 a_4 r_4 + a_4 r_3 + r_1 \end{array}\right).$$

According to Proposition 9 of [14], the stabilizer of $v(0)$ in $GL_2(A)$ is

$$\tilde{\Gamma}_{v(0)} = \{M_2(r_1, r_2) : r_1(-a_3 r_2 + r_1) - a_6 r_2^2 \neq 0\},$$

and of $e(0,0)$ is

$$\tilde{\Gamma}_{e(0,0)} = \{M_4(r_i) : r_1(a_4 r_3 + r_1) + (-a_2 a_4 r_4 + a_1 a_4 r_3 + a_4 r_2 a_1 r_1) a_4 r_4 \neq 0\}.$$

Consider the following identity:

$$\begin{pmatrix} x & -y \\ 0 & 1 \end{pmatrix} M_2(r_1, r_2) \begin{pmatrix} 1/x & y/x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_1 & a_6 r_2 \\ r_2 & r_1 - a_3 r_2 \end{pmatrix} = N_2(r_1, r_2).$$

It follows that

$$\begin{pmatrix} x & -y \\ 0 & 1 \end{pmatrix} \tilde{\Gamma}_{v(0)} \begin{pmatrix} 1/x & y/x \\ 0 & 1 \end{pmatrix} = \{N_2(r_1, r_2) : \det N_2(r_1, r_2) \neq 0\} = \tilde{\Gamma}.$$

Note that the subgroup $\tilde{\Gamma}$ lies in $GL_2(k)$ and that $g = \begin{pmatrix} x & -y \\ 0 & 1 \end{pmatrix}$ is an element of $GL_2(F)$. It follows that $g\Gamma_{v(0)} g^{-1} = \tilde{\Gamma}/k^\times \subset PGL_2(k)$ inside $PGL_2(F)$. Moreover, we can demonstrate the isomorphism of Proposition 4.2 as follows. If $F(0, y) = 0$ has no rational solutions, then define a map $\tilde{\Gamma} \to k(\omega)^\times$ by $N_2(r_1, r_2) \mapsto r_1 + r_2\omega$. One checks easily that this is an isomorphism. If $F(0, y) = 0$ has two solutions, say $u, v \in k$, then it is easy to see that

$$\begin{pmatrix} u & uv \\ \frac{-1}{u(u-v)} & \frac{-1}{u-v} \end{pmatrix} \tilde{\Gamma} \begin{pmatrix} u & uv \\ \frac{-1}{u(u-v)} & \frac{-1}{u-v} \end{pmatrix}^{-1} = D(k)$$

where $D(k) \subset GL_2(k)$ is the subgroup of diagonal matrices. Finally, if $F(0, y) = 0$ has a unique solution, say $u \in k^\times$, then

$$\begin{pmatrix} 0 & -1 \\ 1 & u \end{pmatrix} \tilde{\Gamma} \begin{pmatrix} 0 & -1 \\ 1 & u \end{pmatrix}^{-1} = B(k)$$

where $B(k)$ is the upper triangular subgroup of $GL_2(k)$.

For the group $\Gamma_{e(0,0)}$ we have

$$\begin{pmatrix} x & -y \\ \frac{y}{a_4} & 1-\frac{y^2}{a_4x} \end{pmatrix} \tilde{\Gamma}_{e(0,0)} \begin{pmatrix} x & -y \\ \frac{y}{a_4} & 1-\frac{y^2}{a_4x} \end{pmatrix}^{-1} = GL_2(k)$$

from which it follows that $\Gamma_{e(0,0)}$ is conjugate to $PGL_2(k)$ inside $PGL_2(F)$. (Note that since $\overline{E}$ is smooth, $a_4 \neq 0$.) □

6. Proof of Theorem 1.1

We now prove Theorem 1.1. Corollary 5.2 shows that $H_\bullet(PGL_2(A), \mathbb{Z})$ has image equal to the image of $H_\bullet(PGL_2(k), \mathbb{Z})$ in $H_\bullet(PGL_2(F), \mathbb{Z})$. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & k^\times & \to & GL_2(A) & \to & PGL_2(A) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & F^\times & \to & GL_2(F) & \to & PGL_2(F) & \to & 1 \end{array}$$

and the induced map of Hochschild–Serre spectral sequences

$$\begin{array}{ccccc} E^2_{p,q}(A) & = & H_p(PGL_2(A), H_q(k^\times)) & \Rightarrow & H_{p+q}(GL_2(A), \mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ E^2_{p,q}(F) & = & H_p(PGL_2(F), H_q(F^\times)) & \Rightarrow & H_{p+q}(GL_2(F), \mathbb{Z}). \end{array}$$

Since the extensions are central, the groups $H_q(k^\times)$ (resp. $H_q(F^\times)$) are trivial $PGL_2(A)$ (resp. $PGL_2(F)$) modules. Hence we have the following commutative diagram of universal coefficient sequences

$$\begin{array}{ccccc} H_p(PGL_2(A))\otimes H_q(k^\times) & \to & H_p(PGL_2(A), H_q(k^\times)) & \to & \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(PGL_2(A)), H_q(k^\times)) \\ \downarrow & & \downarrow & & \downarrow \\ H_p(PGL_2(F))\otimes H_q(F^\times) & \to & H_p(PGL_2(F), H_q(F^\times)) & \to & \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(PGL_2(F)), H_q(F^\times)). \end{array}$$

By Corollary 5.2, we see that the image of $E^2_{p,q}(A) \to E^2_{p,q}(F)$ coincides with the image of $E^2_{p,q}(k) \to E^2_{p,q}(F)$. It follows that the same is true of the E^∞ terms:

$$\mathrm{im}\{E^\infty_{p,q}(A) \to E^\infty_{p,q}(F)\} = \mathrm{im}\{E^\infty_{p,q}(k) \to E^\infty_{p,q}(F)\}.$$

Thus, the image of $H_\bullet(GL_2(A), \mathbb{Z})$ in $H_\bullet(GL_2(F), \mathbb{Z})$ coincides with the image of $H_\bullet(GL_2(k), \mathbb{Z})$.

References

[1] *A. Beilinson,* Higher regulators and values of L-functions of curves, Funct. Anal. Appl. **14** (1980), 116–118.

[2] *S. Bloch,* Higher regulators, algebraic K-theory, and zeta functions of elliptic curves, Irvine lecture notes, 1977.

[3] *S. Bloch, D. Grayson,* K_2 and L-functions of elliptic curves: computer calculations, Contemp. Math. **55** (1986), 79–88.

[4] *A. Borel, J. Yang,* The rank conjecture for number fields, Math. Res. Lett. **1** (1994), 689–699.

[5] *K. Brown,* Cohomology of groups, Springer–Verlag, Berlin/Heidelberg/New York, 1982.

[6] *A. Goncharov, A. Levin,* Zagier's conjecture on $L(E, 2)$, preprint.

[7] *K. Knudson,* Integral homology of PGL_2 over elliptic curves, to appear in Algebraic K-theory, Seattle, WA, 1997, W. Raskind and C. Weibel, eds., Proc. Symp. Pure Math.

[8] *J. Milnor, C. Moore,* On the structure of Hopf algebras, Ann. Math. **81** (1965), 211–264.

[9] *Yu. Nesterenko, A. Suslin,* Homology of the full linear group over a local ring, and Milnor's K–theory, Math. USSR Izvestiya **34** (1990), 121–145.

[10] *K. Rolshausen, N. Schappacher,* On the second K-group of an elliptic curve, J. reine angew. Math. **495** (1998), 61–77.

[11] *J. Rosenberg,* Algebraic K-theory and its applications, Springer–Verlag, Berlin/Heidelberg/New York, 1994.

[12] *J.-P. Serre,* Trees, Springer–Verlag, Berlin/ Heidelberg, New York, 1980.

[13] *C. Soulé,* Opérations en K-théorie algébrique, Canad. J. Math. **37** (1985), 488–550.

[14] *S. Takahashi,* The fundamental domain of the tree of $GL(2)$ over the function field of an elliptic curve, Duke Math. J. **73** (1993), 85–97.

[15] *J. Wildeshaus,* On an elliptic analogue of Zagier's conjecture, Duke Math. J. **87** (1997), 355–407.

Department of Mathematics, Northwestern University, Evanston, IL 60208

E-mail address: knudson@math.nwu.edu